

# Log-Sobolev inequalities for semi-direct product operators and applications

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## 1 Introduction and Main Theorem

This paper deals with results of the following type: assume that an operator  $\mathcal{L}_1$  satisfies a log-Sobolev inequality with parameter on a measure space  $(X_1, \mu_1)$ , and that another operator  $\mathcal{L}_0$  satisfies a log-Sobolev inequality (defective or with parameter) on a second measure space  $(X_0, \mu_0)$ . Then, we prove a log-Sobolev inequality (defective or with parameter) for the *semi-direct product operator*

$$\mathcal{L} := \mathcal{L}_0 + N_1^2(x_0)\mathcal{L}_1$$

on the space  $X_0 \times X_1$  with respect to the measure  $\mu = \mu_0 \times \mu_1$ , for any weight function  $N_1 : X_0 \rightarrow \mathbb{R}$  whose set of zeroes is negligible.

More generally, consider  $n + 1$  measure spaces  $(X_i, \mu_i)$ ,  $i = 0, \dots, n$ , let  $X = X_0 \times \dots \times X_n$  be their product with the product measure  $\mu = \mu_0 \otimes \dots \otimes \mu_n$ , and

write  $\tilde{\mu}_i = \mu_0 \otimes \cdots \otimes \mu_{i-1}$ . Let  $\Gamma_i$  be a *carré du champ* on  $X_i$  in the sense of Bakry-Emery [Ba], with domain  $\mathcal{D}(\Gamma_i) \subset L^2(\mu_i)$ . Fix  $n$  real valued weight functions  $N_1, \dots, N_n$  defined on  $X$  with the property that  $N_i$  depends only on the variables  $(x_0, \dots, x_{i-1})$ , nondegenerate in the sense that

$$\tilde{\mu}_i(\{N_i = 0\}) = 0, \quad i = 1, \dots, n. \quad (1.1)$$

Then we set, for any  $f : X \rightarrow \mathbb{R}$ ,  $x = (x_0, \dots, x_n) \in X$ ,

$$\Gamma f(x_0, x_1, \dots, x_n) = \Gamma_0 f(\hat{x}_0)(x_0) + \sum_{i=1}^n N_i^2(x) \Gamma_i f(\hat{x}_i)(x_i).$$

Here we are using the notation

$$\Gamma_i f(\hat{x}_i)(x_i)$$

to mean that the carré du champ  $\Gamma_i$  acts only on the  $i$ -th coordinate, the others remaining fixed. The corresponding operator on  $X$  is defined by

$$\mathcal{L} = \mathcal{L}_0 + \sum_{i=1}^n N_i^2(x) \mathcal{L}_i.$$

We shall call the carré du champ  $\Gamma$  the *semi-direct product* of the champs  $\Gamma_0, \dots, \Gamma_n$ , and  $\mathcal{L}$  the *semi-direct product* of the operators  $\mathcal{L}_i$  associated with the family  $N_i$ .

We also recall that a carré du champ  $\Gamma_0$  is said to satisfy the *diffusion property*, or that it is of *diffusion type*, if, for all functions  $\phi, \chi, \psi$  in an algebra of functions  $\mathcal{A}$  which is dense in its domain  $\mathcal{D}(\Gamma_0)$ , the identity

$$\Gamma_0(\phi\chi, \psi) = \phi\Gamma_0(\chi, \psi) + \chi\Gamma_0(\phi, \psi)$$

is satisfied [Ba].

Our main goal is to prove that if each  $\Gamma_i$  (resp.  $\mathcal{L}_i$ ) for  $i = 0, \dots, n-1$  is of diffusion type and satisfies a log-Sobolev estimate, then their semi-direct product also satisfies suitable generalized log-Sobolev estimates.

We recall the relevant notions. We say that  $(X_i, \mu_i, \mathcal{L}_i)$  satisfies a *log-Sobolev inequality with parameter* (or super log-Sobolev inequality) if there exists a continuous, non increasing function  $M_i : (0, +\infty) \rightarrow \mathbb{R}$  such that

$$(H_i) \quad \forall h \in \mathcal{D}(\Gamma_i), \forall t > 0, \quad \int_{X_i} h^2 \ln \frac{h^2}{\|h\|_2^2} d\mu_i \leq t \int_{X_i} \Gamma_i(h) d\mu_i + M_i(t) \|h\|_2^2 \quad (1.2)$$

with  $\|h\|_2^2 := \|h\|_{L^2(\mu_i)}^2$ . Recall that the relation between the carré  $\Gamma_i$  and the associated operator  $\mathcal{L}_i$  is expressed by

$$\int_{X_i} \Gamma_i(h) d\mu_i = \int_{X_i} \mathcal{L}_i h \cdot h d\mu_i$$

on the domain  $\mathcal{D}(\mathcal{L}_i) \subset \mathcal{D}(\Gamma_i)$ .

We shall say that  $(X_0, \mu_0, \mathcal{L}_0)$  satisfies a defective Gross inequality (or a Gross inequality if  $b = 0$ , see below) if there exists  $a, b > 0$  such that

$$(H'_0) \quad \forall h \in \mathcal{D}(\mathcal{L}_0), \quad \int_{X_0} h^2 \ln \frac{h^2}{\|h\|_{L^2(\mu_0)}^2} d\mu_0 \leq a \int_{X_0} \Gamma_0(h) d\mu_0 + b \|h\|_{L^2(\mu_0)}^2. \quad (1.3)$$

We collect some examples of such inequalities in Section 3.

The main result of the paper is the following

**Theorem 1.1.** *Assume  $X_i, \mu_i, \Gamma_i, \mathcal{L}_i, N_i$  as above satisfy the conditions  $(H_i)$ ,  $i = 0, \dots, n$ . Moreover, assume that  $\Gamma_0, \dots, \Gamma_{n-1}$  have the diffusion property. Then we have:*

1. *For any multiparameter  $t = (t_0, \dots, t_n)$  with  $t_i > 0$ ,  $i = 0, \dots, n$ , the following inequality holds*

$$\int_X h^2 \ln \frac{h^2}{\|h\|_{L^2(\mu)}^2} d\mu \leq \int_X \Gamma^{(t)}(h) d\mu + \int_X [M_0(t_0) + N(t, x)] h^2(x) d\mu(x) \quad (1.4)$$

$$\text{where } \Gamma^{(t)} = t_0 \Gamma_0 + \sum_{i=1}^n t_i N_i^2 \Gamma_i, \quad N(t, x) = \sum_{i=1}^n M_i(t_i N_i^2(x)).$$

2. *Assume that  $(H'_0)$  holds in place of  $(H_0)$ . Then the following inequality holds*

$$\int_X h^2 \ln \frac{h^2}{\|h\|_{L^2(\mu)}^2} d\mu \leq \int_X \tilde{\Gamma}^{(t)}(h) d\mu + \int_X \tilde{W}(t, x) h^2(x) d\mu(x) \quad (1.5)$$

$$\text{where } \tilde{\Gamma}^{(t)} = a \Gamma_0 + \sum_{i=1}^n t_i N_i^2 \Gamma_i, \quad \tilde{W}(t, x) = b + \sum_{i=1}^n M_i(t_i N_i^2(x)).$$

3. *Let  $t = (s, \dots, s) \in \mathbb{R}^{n+1}$ ,  $s > 0$ , and  $\Gamma^{(t)}(h) = s \Gamma(h)$  with  $\Gamma = \Gamma_0 + \sum_{i=1}^n N_i^2 \Gamma_i$ .*

*Then, writing  $\tilde{N}(s, x) = \sum_{i=1}^n M_i(s N_i^2(x))$ , the following inequality holds*

$$\int_X h^2 \ln \frac{h^2}{\|h\|_{L^2(\mu)}^2} d\mu \leq s \int_X \Gamma(h) d\mu + \int_X [M_0(s) + \tilde{N}(s, x)] h^2(x) d\mu(x). \quad (1.6)$$

We notice that the operator  $\mathcal{L}_0$  plays a special role in the previous result, indeed we can assume that  $\mathcal{L}_0$  satisfies either a super log-Sobolev inequality (Statement 1) or a defective Gross inequality (Statement 2). On the other hand, for the remaining operators  $\mathcal{L}_i$  ( $i = 1, \dots, n$ ) we are not able to replace the super log-Sobolev inequality

with a defective Gross inequality, due to the way we use the parameters in the course of the proof. Note that if all the  $\Gamma_i$ 's,  $i = 0, \dots, n$ , are of diffusion type then  $\Gamma^{(t)}$  is also of diffusion type

The proof of Theorem 1.1 is largely inspired by two papers: [CGL] for the method of proof, and [BCL] for the proper formulation of the assumptions. Recall that it is well-known that the logarithmic Sobolev inequality is stable under the usual direct product of spaces endowed with probability measures (the so-called tensorization method), see for instance [ABC]. Our Theorem 1.1 generalizes this situation. The bulk of the proof is contained in Section 2 of the paper, while the following sections are devoted to several examples and applications of our theory. Here, for the convenience of the reader, we would like to illustrate the essential points of the proof in the very special case of the Grushin operator, which can be regarded as a semidirect product in the sense introduced above.

Thus, assume  $X_i = \mathbb{R}_i := \mathbb{R}$  ( $i = 0, 1$ ) with the usual Lebesgue measure which we denote by  $\mu_0 = \mu_1$ , so that  $X = \mathbb{R}^2$  with the 2D Lebesgue measure  $\mu = \mu_0 \otimes \mu_1$ . Consider for  $\alpha > 0$  the operator

$$\mathcal{L}f(x_0, x_1) = \left(\frac{\partial}{\partial x_0}\right)^2 f(x_0, x_1) + |x_0|^{2\alpha} \left(\frac{\partial}{\partial x_1}\right)^2 f(x_0, x_1)$$

with  $\frac{\partial}{\partial x_i}$  the usual partial derivatives. This operator is usually called the *Grushin operator* (but see Section 4 for more general operators with the same structure). We note that  $\mathcal{L}_0 = (\frac{\partial}{\partial x_0})^2$  and  $\mathcal{L}_1 = (\frac{\partial}{\partial x_1})^2$  are standard 1D Laplacians, and hence  $\mathcal{L}$  is not the usual product operator on  $\mathbb{R}^2$ . However  $\mathcal{L}$  can be regarded as the semi-direct product of  $\mathcal{L}_0$ ,  $\mathcal{L}_1$  with function  $N_1(x_0) = |x_0|^\alpha$ , in the sense defined above. We can write the standard super log-Sobolev inequality on  $\mathbb{R}$  (see [C], [BCL]) in the form

$$\forall h \in \mathcal{D}(\mathcal{L}_i), \forall s > 0 \quad \int_{\mathbb{R}_i} h^2 \ln \frac{h^2}{\|h\|_{L^2(\mu_i)}^2} d\mu_i \leq s \int_{\mathbb{R}_i} \Gamma_i(h) d\mu_i + M_i(s) \|h\|_2^2 \quad (1.7)$$

with  $M_i(s) = -\frac{1}{2} \ln(e^2 \pi s)$ ,  $\Gamma_i(h) = |\frac{\partial h}{\partial x_i}|^2$  and  $\|h\|_2^2 = \int_{\mathbb{R}_i} h^2(x_i) d\mu_i(x_i)$ ; this is equivalent to the classical Gross inequality with Gaussian measure ([C], [BCL] and Section 3 below). In particular, we see that our assumptions  $(H_0)$  and  $(H_1)$  are satisfied.

We would like to prove (1.4) in this special situation. To this end, let  $f \in C_0^1(\mathbb{R}^2)$ , let  $t_0, t_1 > 0$ , fix  $x_0 \in \mathbb{R}_0$  and apply (1.7) to the function  $x_1 \mapsto f(x_0, x_1)$  defined on  $X_1$ . We obtain, for all  $s > 0$ ,

$$\int_{\mathbb{R}_1} f^2 \ln f^2 d\mu_1 \leq s \int_{\mathbb{R}_1} \Gamma_1(f) d\mu_1 + M_1(s) \int_{\mathbb{R}_1} f^2(x_0, x_1) d\mu_1(x_1) + \|f\|_{L^2(\mu_1)}^2 \ln \|f\|_{L^2(\mu_1)}^2.$$

For  $x_0 \neq 0$  we can choose  $s = t_1 N_1^2(x_0)$  and integrate with respect to  $\mu_0$ , obtaining

$$\int_{\mathbb{R}^2} f^2 \ln f^2 d\mu \leq t_1 \int_{\mathbb{R}^2} N_1^2 \Gamma_1(f) d\mu + \int_{\mathbb{R}^2} M_1(t_1 N_1^2) f^2 d\mu + \int_{\mathbb{R}_0} h^2(x_0) \ln h^2(x_0) d\mu_0(x_0) \quad (1.8)$$

where  $h(x_0) = \left( \int_{\mathbb{R}_1} f^2(x_0, x_1) d\mu_1(x_1) \right)^{1/2}$ . Now we can apply (1.7) to the function  $h$  and this gives

$$\int_{\mathbb{R}_0} h^2(x_0) \ln h^2(x_0) d\mu_0(x_0) \leq s \int_{\mathbb{R}_0} \Gamma_0(h) d\mu_0 + M_0(s) \int_{\mathbb{R}_0} h^2 d\mu_0 + \|h\|_{L^2(\mu_0)}^2 \ln \|h\|_{L^2(\mu_0)}^2. \quad (1.9)$$

The last term coincides with

$$\|f\|_{L^2(\mu)}^2 \ln \|f\|_{L^2(\mu)}^2$$

while  $\int_{\mathbb{R}_0} h^2(x_0) d\mu_0(x_0) \equiv \|f\|_{L^2(\mu)}^2$ . Now consider the term

$$\int_{\mathbb{R}_0} \Gamma_0(h) d\mu_0 = \int_{\mathbb{R}_0} \left| \frac{\partial h}{\partial x_0} \right|^2 d\mu_0,$$

by the Cauchy-Schwarz inequality we have immediately

$$\begin{aligned} \left| \frac{\partial h}{\partial x_0} \right| &= \left| \frac{1}{2h} \int_{X_1} 2f \frac{\partial f}{\partial x_0} d\mu_1 \right| \leq \frac{1}{h} \left( \int_{X_1} f^2 d\mu_1 \right)^{1/2} \left( \int_{X_1} \left| \frac{\partial f}{\partial x_0} \right|^2 d\mu_1 \right)^{1/2} \\ &= \left( \int_{X_1} \left| \frac{\partial f}{\partial x_0} \right|^2 d\mu_1 \right)^{1/2} \end{aligned}$$

and hence

$$\int_{\mathbb{R}_0} \Gamma_0(h) d\mu_0 \leq \int_{\mathbb{R}^2} \left| \frac{\partial f}{\partial x_0} \right|^2 d\mu = \int_{\mathbb{R}^2} \Gamma_0(f) d\mu.$$

Coming back to (1.8), with  $s = t_0$  in (1.9), we get

$$\begin{aligned} \int_{\mathbb{R}^2} f^2 \ln f^2 d\mu &\leq t_0 \int_{\mathbb{R}^2} \Gamma_0(f) d\mu + t_1 \int_{\mathbb{R}^2} N_1^2 \Gamma_1(f) d\mu + M_0(t_0) \int_{\mathbb{R}^2} f^2 d\mu + \\ &\quad \int_{\mathbb{R}^2} M_1(t_1 N_1^2) f^2 d\mu + \|f\|_{L^2(\mu)}^2 \ln \|f\|_{L^2(\mu)}^2. \end{aligned}$$

Now writing  $t = (t_0, t_1)$ ,  $\Gamma^{(t)} = t_0 \Gamma_0 + t_1 N_1^2 \Gamma_1$  and  $N(t, x) = M_0(t_0) + M_1(t_1 N_1^2(x))$ , the last inequality can be written as

$$\int_{\mathbb{R}^2} f^2 \ln f^2 d\mu \leq \int_{\mathbb{R}^2} \Gamma^{(t)}(f) d\mu + \int_{\mathbb{R}^2} N(t, x) f^2(x) d\mu(x) + \|f\|_{L^2(\mu)}^2 \ln \|f\|_{L^2(\mu)}^2$$

which is exactly (1.4) for our choice of operators and spaces.

The plan of the paper is the following. Section 2 is devoted to the proof of the main result Theorem 1.1. In Section 3 we collect a few examples of super log-Sobolev inequalities arising in different contexts, from geometry, mathematical physics and the general theory of PDEs. These examples are further discussed in Section 4 where we apply our main result to obtain explicit inequalities in several specific

cases. Finally, Section 5 contains some ultracontractive (i.e. heat kernel) bounds which can be deduced in some special cases from our inequalities. The paper is concluded with a technical Appendix where we prove two Hardy type inequalities in the spirit of the assumption of Rosen's lemma [D, Eq.4.4.2], which are necessary for the proof of the ultracontractive bounds of Section 5.

## 2 Proof of the Main Theorem

We begin by recalling our notations. Let  $(X_i, \mu_i)$  for  $i = 0, \dots, n$  be measure spaces,  $X = X_0 \times \dots \times X_n$  their product and  $\mu = \mu_0 \otimes \dots \otimes \mu_n$  the product measure on  $X$ . We also denote by  $\tilde{\mu}_i = \mu_0 \otimes \dots \otimes \mu_{i-1}$ ,  $\bar{\mu}_i = \mu_i \otimes \dots \otimes \mu_n$ ,  $\bar{X}_i = X_i \times \dots \times X_n$ ,  $\tilde{\mu}_{i,j} = \mu_i \otimes \dots \otimes \mu_j$  and  $\tilde{X}_{i,j} = X_i \times \dots \times X_j$  for  $0 \leq i, j \leq n$ .

We denote by  $\Gamma_i$  a *carré du champ* defined on each  $X_i$ , with domain  $\mathcal{D}(\Gamma_i) \subset L^2(\mu_i)$  (see [Ba]); moreover, we fix  $n$  real valued weight functions  $N_1, \dots, N_n$  on  $X$  with the property that  $N_i$  depends only on the variables  $(x_0, \dots, x_{i-1})$ , nondegenerate in the sense that  $\tilde{\mu}_i(\{N_i = 0\}) = 0$  for all  $i = 1, \dots, n$ . Then we set, for any  $f : X \rightarrow \mathbb{R}$ ,

$$\Gamma f(x_0, x_1, \dots, x_n) = \Gamma_0 f(\hat{x}_0)(x_0) + \sum_{i=1}^n N_i^2(x) \Gamma_i f(\hat{x}_i)(x_i).$$

Here we are using the notation

$$\Gamma_i f(\hat{x}_i)(x_i)$$

to mean that the carré du champ  $\Gamma_i$  acts only on the  $i$ -th coordinate, the others remaining fixed. For brevity we shall write simply

$$\Gamma f = \Gamma_0 f + \sum_{i=1}^n N_i^2 \Gamma_i f.$$

In the same way, the corresponding operator on  $X$  is defined by

$$\mathcal{L} f = \mathcal{L}_0 f + \sum_{i=1}^n N_i^2 \mathcal{L}_i f$$

(where as before  $\mathcal{L}_i$  acts only on the variable  $x_i$  while the others remain fixed).

We now state a useful lemma, which extends an analogous result proved in [CGL, p.99] for sums of squares of vector fields or second order differential operators without constant term.

**Lemma 2.1.** *Given  $f \in \mathcal{D}(\mathcal{L})$ , denote by  $h_{n-k}$ ,  $k = 1, \dots, n$  the functions*

$$h_{n-k}(x_0, \dots, x_{n-k})^2 = \int_{\bar{X}_{n-k+1}} f(x_0, x_1, \dots, x_n)^2 d\bar{\mu}_{n-k+1}.$$

Assume that  $\Gamma_0, \dots, \Gamma_{n-1}$  are carré du champ with the diffusion property. Then we have

$$\int_{X_{n-k}} \Gamma_{n-k}(h_{n-k}) d\mu_{n-k} \leq \int_{\bar{X}_{n-k}} \Gamma_{n-k}(f) d\bar{\mu}_{n-k}.$$

(where both sides are functions of the variables  $(x_0, x_1, \dots, x_{n-k-1})$  only).

**Proof.** We shall write the details of the proof in the case  $n = 1$  only; the general case is completely analogous.

We recall the relation between the carré du champ  $\Gamma_0$  and the corresponding generator  $\mathcal{L}_0$ : one introduces the bilinear form

$$\Gamma_0(\phi, \psi) = \frac{1}{2} [\mathcal{L}_0(\phi\psi) - \phi\mathcal{L}_0(\psi) - \psi\mathcal{L}_0(\phi)]$$

and then the relation is given by

$$\Gamma_0(\phi) = \Gamma_0(\phi, \phi).$$

We must prove the inequality

$$\int_{X_0} \Gamma_0(h) d\mu_0 \leq \int_{X_0 \times X_1} \Gamma_0(f) d\mu_0 d\mu_1 \quad (2.10)$$

where  $h$  and  $f$  are related by

$$h(x_0) = \left( \int_{X_1} f(x_0, x_1)^2 d\mu_1(x_1) \right)^{1/2}.$$

We can write

$$\Gamma_0(h^2) = \Gamma_0(h^2, h^2) = \Gamma_0(h^2, \int f^2 d\mu_1) = \int \Gamma_0(h^2, f^2) d\mu_1$$

by linearity. Then using the diffusion property

$$\Gamma_0(\phi\chi, \psi) = \phi\Gamma_0(\chi, \psi) + \chi\Gamma_0(\phi, \psi)$$

and by Cauchy-Schwarz inequality in  $L^2$ , we have

$$\Gamma_0(h^2) = 2 \int f \Gamma_0(h^2, f) d\mu_1 \leq 2h \left( \int \Gamma_0(h^2, f)^2 d\mu_1 \right)^{1/2}.$$

Using now the Cauchy-Schwarz inequality for the bilinear form  $\Gamma_0$ , we deduce

$$\leq 2h \left( \int \Gamma_0(h^2) \Gamma_0(f) d\mu_1 \right)^{1/2} = 2h \Gamma_0(h^2)^{1/2} \left( \int \Gamma_0(f) d\mu_1 \right)^{1/2}.$$

Thus we have proved the inequality

$$\Gamma_0(h^2) \leq 4h^2 \int \Gamma_0(f) d\mu_1. \quad (2.11)$$

Now we notice that, again by the diffusion property,

$$\Gamma_0(h^2) = 4h^2\Gamma_0(h)$$

and together with (2.11) this implies (2.10).

We fix  $n \geq 1$  and prove the main theorem by induction on  $k = 1, \dots, n$ . Let  $t_i > 0$  for  $i = 0, \dots, n$ ; we must prove the inequality

$$(R_k) \quad \int_{\overline{X}_{n-k}} f^2 \ln f^2 d\overline{\mu}_{n-k} \leq \int_{\overline{X}_{n-k}} \sum_{i=n-k+1}^n t_i N_i^2 \Gamma_i(f) d\overline{\mu}_{n-k} \\ + \int_{\overline{X}_{n-k}} \sum_{i=n-k+1}^n M_i(t_i N_i^2) f^2 d\overline{\mu}_{n-k} + \int_{X_{n-k}} h_{n-k}^2 \ln h_{n-k}^2 d\mu_{n-k}.$$

Notice that both sides depend only on the variables  $(x_0, x_1, \dots, x_{n-k-1})$ ; here and in the following, an integral like  $\int_{\overline{X}_{n-k}} f d\overline{\mu}_{n-k}$  denotes integration of the function  $f(x_0, x_1, \dots, x_n)$  with respect to the set of variables  $(x_{n-k}, x_{n-k+1}, \dots, x_n)$  in the measure  $d\overline{\mu}_{n-k}$ .

*Step 1:* We start by proving  $(R_1)$  i.e.

$$\int_{X_{n-1} \times X_n} f^2 \ln f^2 d\mu_{n-1} d\mu_n \leq \int_{X_{n-1} \times X_n} t_n N_n^2 \Gamma_n(f) d\mu_{n-1} d\mu_n + \\ \int_{X_{n-1} \times X_n} M_n(t_n N_n^2) f^2 d\mu_{n-1} d\mu_n + \int_{X_{n-1}} h_{n-1}^2 \ln h_{n-1}^2 d\mu_{n-1}.$$

In order to prove this, we apply  $(H_n)$  to the function  $x_n \in X_n \rightarrow f(x_0, x_1, \dots, x_n)$ . Recalling the definition of  $h_{n-1}$  above, we obtain for any  $s > 0$

$$\int_{X_n} f^2 \ln f^2 d\mu_n \leq s \int_{X_n} \Gamma_n(f) d\mu_n + \int_{X_n} M_n(s) f^2 d\mu_n + h_{n-1}^2 \ln h_{n-1}^2.$$

We can choose now  $s = t_n N_n^2(x) + \epsilon$ ,  $\epsilon > 0$  (recall that  $N_n$  does not depend on  $x_n$ ) and integrate w.r.to  $\mu_{n-1}$ ; since  $M_n(s)$  is non increasing, by letting  $\epsilon \rightarrow 0$  we obtain  $(R_1)$ .

*Step 2:* We now assume that  $(R_k)$  is true for some  $k$ ,  $1 \leq k \leq n-1$ , and deduce  $(R_{k+1})$ . First of all, we deal with the last term of the inequality  $(R_k)$  i.e.

$$\int_{X_{n-k}} h_{n-k}^2 \ln h_{n-k}^2 d\mu_{n-k}.$$

If we apply assumption  $(H_{n-k})$  to the function  $x_{n-k} \in X_{n-k} \rightarrow h_{n-k}(x_0, x_1, \dots, x_{n-k})$ , we obtain, for any  $s > 0$ ,

$$\int_{X_{n-k}} h_{n-k}^2 \ln h_{n-k}^2 d\mu_{n-k} \leq s \int_{X_{n-k}} \Gamma_{n-k}(h_{n-k}) d\mu_{n-k} + \int_{X_{n-k}} M_{n-k}(s) h_{n-k}^2 d\mu_{n-k}$$



$$+ h_{n-k-1}^2 \ln h_{n-k-1}^2.$$

Indeed,  $h_{n-k-1}^2 = \int_{X_{n-k}} h_{n-k}^2 d\mu_{n-k}$ . Then we choose  $s = t_{n-k} N_{n-k}^2(x) + \epsilon$ ,  $\epsilon > 0$ , which is possible since  $N_{n-k}$  does not depend of the variables  $(x_{n-k}, \dots, x_n)$ . Applying Lemma 2.1 to the first term at the right hand side of the previous inequality, integrating the inequality w.r.to  $d\mu_{n-k-1}$  over  $X_{n-k-1}$ , and letting  $\epsilon \rightarrow 0$  as above, we obtain

$$\begin{aligned} \int_{\tilde{X}_{n-k-1, n-k}} h_{n-k}^2 \ln h_{n-k}^2 d\tilde{\mu}_{n-k-1, n-k} &\leq \int_{\bar{X}_{n-k-1}} t_{n-k} N_{n-k}^2 \Gamma_{n-k}(f) d\bar{\mu}_{n-k-1} + \\ &\int_{\bar{X}_{n-k-1}} M_{n-k}(t_{n-k} N_{n-k}^2) f^2 d\bar{\mu}_{n-k-1} + \int_{X_{n-k-1}} h_{n-k-1}^2 \ln h_{n-k-1}^2 d\mu_{n-k-1}. \end{aligned} \quad (2.12)$$

On the other hand, if we integrate the inequality  $(R_k)$  w.r.to  $d\mu_{n-k-1}$  we get

$$\begin{aligned} \int_{\bar{X}_{n-k-1}} f^2 \ln f^2 d\bar{\mu}_{n-k-1} &\leq \int_{\bar{X}_{n-k-1}} \sum_{i=n-k+1}^n t_i N_i^2 \Gamma_i(f) d\bar{\mu}_{n-k-1} + \\ &\int_{\bar{X}_{n-k-1}} \sum_{i=n-k+1}^n M_i(t_i N_i^2) f^2 d\bar{\mu}_{n-k-1} + \int_{\tilde{X}_{n-k-1, n-k}} h_{n-k}^2 \ln h_{n-k}^2 d\tilde{\mu}_{n-k-1, n-k}. \end{aligned} \quad (2.13)$$

Applying (2.12) to estimate the last term in (2.13) we finally deduce  $(R_{k+1})$ , and this concludes the induction step.

We are now ready to prove Statements 1 and 2 of Theorem 1.1.

For Statement 1, we apply  $(R_k)$  with  $k = n$  for  $n \geq 1$ . Then our conclusion (1.4) differs from  $(R_n)$  only by the expression

$$\int_{X_0} h_0^2 \ln h_0^2 d\mu_0.$$

This term is treated with the assumption  $(H_0)$ . Indeed, using Lemma 2.1, we can write

$$\begin{aligned} \int_{X_0} h_0^2 \ln h_0^2 d\mu_0 &\leq t_0 \int_{X_0} \Gamma_0(h_0) d\mu_0 + M_0(t_0) \int_{X_0} h_0^2 d\mu_0 \\ &\leq t_0 \int_X \Gamma_0(f) d\mu + M_0(t_0) \int_X f^2 d\mu \end{aligned}$$

and Statement 1 follows.

Statement 2 can be proved exactly in the same way; indeed, it is sufficient to notice that the variable  $t_0$  and  $M(t_0)$  can be considered as two constants  $a > 0$  and  $b \geq 0$  (also  $b \in \mathbb{R}$  can be considered), so that  $(H'_0)$  can be used in place of  $(H_0)$ . Statements 3 is just a special case of Statement 1. The proof is completed.

### 3 Examples of Logarithmic Sobolev inequality

In this section, we present several concrete situations where either a Gross inequality of classical or defective type, or a logarithmic Sobolev inequality with parameter are satisfied. Most situations are essentially known, but in some cases we present extensions of known results which are not completely standard. These examples can be used as building blocks and combined to obtain a variety of *semiproduct operators* to which our general theory applies. Recall that on any measure space  $(X, \mu)$  the *entropy* of a non negative measurable function  $f$  is defined by

$$\mathbf{Ent}_\mu(f) := \int_X f \ln f \, d\mu.$$

#### 3.1 Gross type inequalities

We begin by recalling a few important situations where a Gross type inequality is satisfied.

1. The basic example is given by  $X = \mathbb{R}^n$  with the standard Gaussian measure  $d\mu(x) = d\gamma_n(x) = (2\pi)^{-n/2} e^{-\frac{|x|^2}{2}}$  where  $dx$  is the Lebesgue measure. The classical Gross inequality is the following

$$\|f\|_2 = 1 \quad \implies \quad \mathbf{Ent}_{\gamma_n}(f^2) \leq c \int_{\mathbb{R}^n} |\nabla f|^2 \, d\gamma_n, \quad c = 2 \quad (3.14)$$

where  $|\nabla f|^2 = \sum_{i=1}^n \left| \frac{\partial f}{\partial x_i} \right|^2$ . An essential feature of the inequality is that the constant  $c = 2$  does not depend on the space dimension  $n$ . Note also that the inequality is sharp, see [GR1]. See also inequalities (3.18) and (3.20) in Section 3.2 below, developed from the Gross inequality for an infinite measure and the uniform measure on an open set  $\Omega \subset \mathbb{R}^n$  of finite measure.

2. An analogous result holds on the  $n$ -dimensional torus  $\mathbb{T}^n = [-\pi, \pi]^n$ . Denoting by  $\mu_n$  the normalized uniform measure on  $\mathbb{T}^n$ , we have for all  $f \in C^1(\mathbb{T}^n)$  (i.e.  $f \in C^1(\mathbb{R}^n)$  and  $2\pi$ -periodic in each variable)

$$\|f\|_2 = 1 \quad \implies \quad \mathbf{Ent}_{\mu_n}(f^2) \leq c \int_{\mathbb{T}^n} |\nabla f|^2 \, d\mu_n \quad c = 2. \quad (3.15)$$

For  $n = 1$ , the inequality is sharp, see [We] (see also [EY] for a simple proof). Since the constant  $c = 2$  is independent of the dimension  $n$ , the general case  $n \geq 1$  is obtained by the classical argument of tensorization. This inequality (3.15) also extends to the infinite dimensional case. More precisely, denote by  $\mathbb{T}^\infty$  the infinite dimensional torus with Haar probability measure  $\mu^\mathbb{N}$  (see [Ben] for more details and in particular a study of several heat kernels on  $\mathbb{T}^\infty$ ). Then the above Gross type inequality is valid for all *cylindrical* functions in  $C^1(\mathbb{T}^\infty)$ , i.e., depending on a finite number of coordinates.

3. Consider for some  $L > 0$  an interval  $X = [0, L] \subset \mathbb{R}$  endowed with the uniform measure  $d\mu(x) = \frac{dx}{L}$ ; no periodic conditions at the boundary are imposed. Then we have

$$\|f\|_2 = 1 \implies \mathbf{Ent}_\mu(f^2) \leq \frac{2L^2}{\pi^2} \int_{[0,L]} |\nabla f|^2 d\mu$$

(see [EY], [G] for this and related results). Compare this result with the case of periodic boundary conditions, where we have

$$\|f\|_2 = 1 \implies \mathbf{Ent}_\mu(f^2) \leq \frac{L^2}{2\pi^2} \int_{[0,L]} |\nabla f|^2 d\mu.$$

When  $L = 2\pi$  this is contained in (3.15).

4. The case of an interval  $I = (a, b) \subset \mathbb{R}$  and a general weighted Lebesgue measure  $d\mu(x) = \frac{1}{Z} e^{-V(x)} dx$ , where  $Z = \int_I e^{-V(x)} dx$ , is studied in [L, Section 7]. Then the following results are valid:

- (a) When  $I = (0, 2\pi)$  and  $V(x) = -2\gamma \ln \sin(\frac{x}{2})$  with  $\gamma > 1/2$ ,

$$\|f\|_2 = 1 \implies \mathbf{Ent}_\mu(f^2) \leq \frac{8}{1+2\gamma} \int_I |\nabla f|^2 d\mu.$$

- (b) When  $I = (-1, 1)$  and  $V(x) = -2\alpha \ln(1-x) - 2\beta \ln(1+x)$ , with

$$\gamma = \min(\alpha, \beta), \quad \delta = \max(\alpha, \beta),$$

if  $\gamma > 1/2$  then

$$\|f\|_2 = 1 \implies \mathbf{Ent}_\mu(f^2) \leq \frac{\delta}{\gamma(1+2\delta)} \int_I |\nabla f|^2 d\mu.$$

5. Let  $\lambda > -\frac{1}{2}$  and the probability measure  $d\mu_\lambda(x) = A_\lambda(1-x^2)^{\lambda-(1/2)} dx$  on  $I = [-1, 1]$ . Then for any real-valued  $f \in C^2([-1, 1])$ , we have the following sharp log-Sobolev inequality

$$\|f\|_2 = 1 \implies \mathbf{Ent}_{\mu_\lambda}(f^2) \leq \frac{2}{2\lambda+1} \int_I H_\lambda f(x) f(x) d\mu_\lambda(x),$$

with the infinitesimal generator is given by  $H_\lambda f(x) = -(1-x^2)f''(x) + (2\lambda+1)xf(x)$ ,  $x \in I$ , (see [MW, Th.1, p.268]. Here the "gradient" has the following form  $\nabla_1 f(x) = \sqrt{1-x^2} \cdot f'(x) = \sqrt{1-x^2} \cdot \nabla f(x)$  and

$$\int_I H_\lambda f(x) f(x) d\mu_\lambda(x) = \int_I |\nabla_1 f(x)|^2 d\mu_\lambda(x)$$

(compare with 4.(b)).

6. Similar inequalities can be proved for some classes of manifolds, both compact and non compact; here we prefer to skip this line of research and refer the reader to the survey [GR2] and the book by F-Y.Wang [Wa1, Example 5.7.2]).

### 3.2 Logarithmic Sobolev inequalities with parameter

We now examine a few cases where a logarithmic Sobolev inequality with parameter is known to hold. Notice that the following results are partially new (although they can be proved by suitable extension of the standard techniques).

1. The simplest logarithmic Sobolev inequality with parameter corresponds to the choice  $X = \mathbb{R}^n$  and  $\mathcal{L} = \Delta$ , the usual (positive) Laplacian. Then for any  $t > 0$  we have

$$\int_{\mathbb{R}^n} f^2 \ln \frac{f^2}{\|f\|_2^2} dx \leq t \int_{\mathbb{R}^n} \Delta f \cdot f dx + M(t) \|f\|_2^2 \quad (3.16)$$

where  $M(t) = -\frac{n}{2} \ln(\pi e^2 t)$  and  $dx$  is the Lebesgue measure. Notice that we impose no constraint on  $\|f\|_2$ . The inequality is sharp and the extremal functions are Gaussian functions of the form  $p_t(x) = (2\pi t)^{-n/2} \exp(-\frac{|x-a|^2}{2t})$ ,  $a \in \mathbb{R}^n$  (see [C], [BCL]). We shall refer to (3.16) as a “flat” Gross inequality, to emphasize that the supporting space  $\mathbb{R}^n$  is viewed as a manifold with zero curvature, as opposed to more general inequalities on more general, nontrivial manifolds. Important consequences can be drawn from (3.16), in particularity it can be extended to more general measures of gaussian type (see [BCL] for some examples). As an application of our theory, we shall extend (3.16) to operators of Grushin type.

We recall that inequality (3.16) is actually equivalent to the Gross inequality (3.14), as it can be proved via a simple argument. For instance, it is possible to deduce (3.16) from Gross in two steps. First of all, we apply (3.14) to a function  $g$  with weighted  $L^2$  norm  $\|g\|_2 = 1$ , and defining  $f$  via  $f^2 = (2\pi)^{-n/2} e^{-\frac{|x|^2}{2}} g^2$  we obtain

$$\int_{\mathbb{R}^n} f^2 \ln f^2 dx \leq 2 \int_{\mathbb{R}^n} |\nabla f|^2 dx - n - \frac{n}{2} \ln(2\pi).$$

Notice also that the unweighted  $L^2$  norm of  $f$  is  $\|f\|_2 = 1$ . For the second step, we perform a dilation  $h_\lambda(x) = \lambda x$  with  $\lambda > 0$ : since  $\Delta$  satisfies  $\Delta(f \circ h_\lambda) = \lambda^2(\Delta f) \circ h_\lambda$  changing variables in the integrals we obtain

$$\int_{\mathbb{R}^n} f^2 \ln f^2 dx \leq 2\lambda^2 \int_{\mathbb{R}^n} \Delta f \cdot f dx - \frac{n}{2} \ln(2\pi e^2 \lambda^2).$$

It is now sufficient to set  $t = 2\lambda^2$  and rescale the norm of  $f$  to obtain, for any function  $f$ ,

$$\int_{\mathbb{R}^n} f^2 \ln \frac{f^2}{\|f\|_2^2} dx \leq t \int_{\mathbb{R}^n} \Delta f \cdot f dx - \frac{n}{2} \ln(\pi e^2 t) \|f\|_2^2 \quad (3.17)$$

which is precisely (3.16). As a side remark we observe that the log-Sobolev inequality (3.16) is stable by the dilation structure imposed by the operator,

in this case the Laplacian on  $\mathbb{R}^n$ . We shall encounter a similar situation in the case of Grushin operators (see Example 1 in Section 4).

Notice also that, choosing  $t = (\pi e^2)^{-1}$  in (3.17), we deduce that the log-Sobolev inequality of Gross type is satisfied for an *infinite* measure i.e. the Lebesgue measure on  $\mathbb{R}^n$ :

$$\int_{\mathbb{R}^n} f^2 \ln \frac{f^2}{\|f\|_2^2} dx \leq \frac{1}{\pi e^2} \int_{\mathbb{R}^n} \Delta f \cdot f dx. \quad (3.18)$$

Here again, the inequality (3.18) implies (3.17) by dilation.

2. When  $X = \Omega$  is an open set of  $\mathbb{R}^n$  with Lebesgue finite measure  $0 < |\Omega| < \infty$ , we easily deduce by restriction the super log-Sobolev inequality on  $\Omega$  for smooth function with compact support on  $\Omega$  from super log-Sobolev inequality on the whole Euclidean space. This formulation corresponds to the Dirichlet problem on  $\Omega$  for the Laplacian. Indeed, for any  $f \in C_0^\infty(\Omega)$  we have

$$\int_{\Omega} f^2 \ln \frac{f^2}{\|f\|_2^2} d\mu \leq t \int_{\Omega} \Delta f \cdot f d\mu + \ln(|\Omega|(\pi e^2 t)^{-\frac{n}{2}}) \|f\|_2^2 \quad (3.19)$$

where  $\mu$  is the normalized Lebesgue measure on  $\Omega$  and  $\|f\|_2^2 = \|f\|_{L^2(\mu)}^2$ . To prove (3.19) it is sufficient to take a smooth function  $f$  with compact support on  $\Omega$  and apply (3.17) to  $f/\sqrt{|\Omega|}$ .

Moreover, as in the first example, we deduce a log-Sobolev inequality of Gross type on  $\Omega$  for the probability measure  $\mu$  by setting  $t = (\pi e^2)^{-1} |\Omega|^{2/n}$  in (3.19). We get

$$\int_{\Omega} f^2 \ln \frac{f^2}{\|f\|_2^2} d\mu \leq (\pi e^2)^{-1} |\Omega|^{2/n} \int_{\Omega} \Delta f \cdot f d\mu. \quad (3.20)$$

3. An important extension of the previous inequalities can be obtained in the case of Lie groups of polynomial growth (up to optimality). Let  $G$  be such a group with global dimension  $D$ , and consider a Hörmander system of left-invariant vector fields  $X = (X_1, X_2, \dots, X_m)$  on  $G$ . We denote by  $L$  the sub-Laplacian  $L = -\sum_{i=1}^m X_i^2$  and by  $d$  the local dimension associated with  $X$ . We assume that  $d \leq D$ . Then the semigroup  $(T_t)$  generated by  $L$  satisfies

$$\|T_t\|_{2 \rightarrow \infty} \leq c_0 t^{-n/4}$$

for any  $t > 0$  and any  $n \in [d, D]$  (see [VSC]) ( $\|T\|_{p \rightarrow q}$  denotes the norm of the operator  $T : L^p \rightarrow L^q$ ). By Theorem 2.2.3 in [D], we obtain the logarithmic Sobolev inequality with parameter

$$\int_G f^2 \ln \frac{f^2}{\|f\|_2^2} dx \leq t \int_G \mathcal{L} f \cdot f dx + M(t) \|f\|_2^2$$

where  $M(t) = 2 \ln(c_0) - \frac{n}{2} \ln(t/2)$  and  $dx$  is the bi-invariant Haar measure. Notice that, in the non compact case, the parameter appears in a natural way, differently from the compact case. In this context, the log-Sobolev inequalities (3.18) and (3.20) hold true with appropriate constants.

4. A further extension concerns the case of Lie groups of exponential growth. We shall consider here *Damek-Ricci spaces*, also known as *harmonic NA groups* ([ADY, DR]); these solvable Lie groups include all symmetric spaces of non compact type and rank one. We briefly recall the definition of the spaces. Let  $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$  be an Heisenberg type algebra and let  $N$  be the connected and simply connected Lie group associated with  $\mathfrak{n}$ . Let  $S$  be the one-dimensional extension of  $N$  obtained by making  $A = \mathbb{R}^+$  act on  $N$  by homogeneous dilations. We denote by  $Q$  the homogeneous dimension of  $N$  and by  $n$  the dimension of  $S$ . Let  $H$  denote a vector in  $\mathfrak{a}$  acting on  $\mathfrak{n}$  with eigenvalues  $1/2$  and (possibly)  $1$ ; we extend the inner product on  $\mathfrak{n}$  to the algebra  $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$ , by requiring  $\mathfrak{n}$  and  $\mathfrak{a}$  to be orthogonal and  $H$  to be a unit vector. We denote by  $d$  the left-invariant distance on  $S$  associated with the Riemannian metric on  $S$  which agrees with the inner product on  $\mathfrak{s}$  at the identity. The Riemannian manifold  $(S, d)$  is usually referred to as *Damek-Ricci space*.

Note that  $S$  is nonunimodular in general; denote by  $\lambda$  and  $\rho$  the left and right Haar measures on  $S$ , respectively. It is well known that the spaces  $(S, d, \lambda)$  and  $(S, d, \rho)$  are of *exponential growth*. In particular, the two following Laplacians on  $S$  have been the object of investigation :

- (i) The Laplace-Beltrami operator  $\Delta_S$  associated with the Riemannian metric  $d$ . The operator  $-\Delta_S$  is left-invariant, it is essentially self-adjoint on  $L^2(S, \lambda)$  and its spectrum is the half line  $[Q^2/4, \infty)$ .
- (ii) The left-invariant Laplacian  $\mathcal{L} = \sum_{i=0}^{n-1} Y_i^2$ , where  $Y_0, \dots, Y_{n-1}$  are left-invariant vector fields such that at the identity  $Y_0 = H$ ,  $\{Y_1, \dots, Y_{m_v}\}$  is an orthonormal basis of  $\mathfrak{v}$  and  $\{Y_{m_v+1}, \dots, Y_{n-1}\}$  is an orthonormal basis of  $\mathfrak{z}$ . The operator  $-\mathcal{L}$  is essentially self-adjoint on  $L^2(S, \rho)$  and its spectrum is  $[0, \infty)$ .

In case (ii) the theory of heat kernels is still under development, and we shall focus here on the case (i) of the Laplace-Beltrami operator  $-\Delta_S$  on a Damek-Ricci space  $S$ . This operator was studied in [APV]; in particular, the following pointwise estimate on the heat kernel  $h_t$  corresponding to the heat operator  $e^{t\Delta_S}$ ,  $t > 0$  was proved in Proposition 3.1 [APV]: there exists a positive constant  $C$  such that, for every  $t > 0$  and for any  $r \in \mathbb{R}^+$ , we have

$$0 < h_t(r) \leq \begin{cases} C t^{-n/2} (1+r)^{\frac{n-1}{2}} e^{-\frac{Q}{2}r} e^{-\frac{1}{4}\{Q^2t + \frac{r^2}{t}\}} & \text{if } t \leq 1+r, \\ C t^{-3/2} (1+r) e^{-\frac{Q}{2}r} e^{-\frac{1}{4}\{Q^2t + \frac{r^2}{t}\}} & \text{if } t > 1+r. \end{cases} \quad (3.21)$$

Here  $r$  denotes the radial variable on  $S$ . As a consequence we have

$$\|e^{t\Delta_S}\|_{1 \rightarrow \infty} \leq \begin{cases} C t^{-n/2} e^{-\frac{Q^2t}{4}} & \text{if } 0 < t \leq 1, \\ C t^{-3/2} e^{-\frac{Q^2t}{4}} & \text{if } t > 1, \end{cases}$$

which implies

$$\|e^{t\Delta_S}\|_{2 \rightarrow \infty} \leq \begin{cases} \sqrt{C} t^{-n/4} e^{-\frac{Q^2 t}{8}} & \text{if } 0 < t \leq 1, \\ \sqrt{C} t^{-3/4} e^{-\frac{Q^2 t}{8}} & \text{if } t > 1. \end{cases}$$

Thus we can apply again Davies' result and we obtain the following logarithmic Sobolev inequality with parameter

$$\int_S f^2 \ln \frac{f^2}{\|f\|_2^2} dx \leq t(-\Delta_S f, f)_{L^2} + M(t) \|f\|_2^2 \quad (3.22)$$

with

$$M(t) = \begin{cases} \ln(C 2^{n/2}) - \frac{n}{2} \ln t - \frac{Q^2 t}{8} & \text{if } 0 < t \leq 1, \\ \ln(C 2^{3/2}) - \frac{3}{2} \ln t - \frac{Q^2 t}{8} & \text{if } t > 1. \end{cases}$$

5. Our last example is taken from the theory of Schrödinger operators. Consider on  $\mathbb{R}^n$  the *electromagnetic Schrödinger operator*

$$H = -(\nabla - iA(x))^2 + V(x).$$

Then it is possible to prove, under very general assumptions on the potentials  $V, A$ , that the heat kernel  $e^{tH}(x, y)$  satisfies a pointwise gaussian estimate for all times. In order to give the precise assumptions on the coefficients, we need to recall the definition of Kato class and Kato norm:

**Definition 3.1.** *The measurable function  $V(x)$  on  $\mathbb{R}^n$ ,  $n \geq 3$ , is said to belong to the Kato class if*

$$\lim_{r \downarrow 0} \sup_{x \in \mathbb{R}^n} \int_{|x-y| < r} \frac{|V(y)|}{|x-y|^{n-2}} dy = 0. \quad (3.23)$$

Moreover, the Kato norm of  $V(x)$  is defined as

$$\|V\|_K = \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|V(y)|}{|x-y|^{n-2}} dy. \quad (3.24)$$

For  $n = 2$  the kernel  $|x-y|^{2-n}$  is replaced by  $\ln(|x-y|^{-1})$ .

Then our pointwise gaussian estimate is the following

**Proposition 3.2.** *Consider the self-adjoint operator  $H = -(\nabla - iA(x))^2 + V(x)$  on  $L^2(\mathbb{R}^n)$ ,  $n \geq 3$ . Assume that  $A \in L^2_{loc}(\mathbb{R}^n, \mathbb{R}^n)$ , moreover  $V$  is real valued and the positive and negative parts  $V_{\pm}$  of  $V$  satisfy*

$$V_+ \text{ is of Kato class,} \quad (3.25)$$

$$\|V_-\|_K < c_n = \pi^{n/2} / \Gamma\left(\frac{n}{2} - 1\right). \quad (3.26)$$

Then  $e^{-tH}$  is an integral operator and its heat kernel  $p_t(x, y)$  satisfies the pointwise estimate

$$|p_t(x, y)| \leq \frac{(2\pi t)^{-n/2}}{1 - \|V_-\|_K / c_n} e^{-|x-y|^2/(8t)}. \quad (3.27)$$

From estimate (3.27) we obtain

$$\|e^{-tH}\|_{1 \rightarrow \infty} \leq \frac{(2\pi t)^{-n/2}}{1 - \|V_-\|_K/c_n}$$

and as a consequence

$$\|e^{-tH}\|_{2 \rightarrow \infty} \leq \frac{(2\pi t)^{-n/4}}{(1 - \|V_-\|_K/c_n)^{1/2}}$$

Again by Davies' result we obtain the logarithmic Sobolev inequality with parameter

$$\int_{\mathbb{R}^n} f^2 \ln \frac{f^2}{\|f\|_2^2} dx \leq t(Hf, f)_{L^2} + M(t)\|f\|_2^2 \quad (3.28)$$

with

$$M(t) = \ln \left( \frac{(\pi t)^{-n/2}}{1 - \|V_-\|_K/c_n} \right).$$

We now give a concise but complete proof of the heat kernel estimate in Proposition 3.2.

*Proof.* Assume first that the magnetic part is zero:  $A \equiv 0$ . Then the estimate (3.27) under the assumptions (3.25), (3.26) was proved in the paper [DP] (see the second part of Proposition 5.1).

Consider now the case the magnetic part is different from zero. We recall Simon's diamagnetic pointwise inequality (see e.g. Theorem B.13.2 in [S]), which holds under the assumption  $A \in L_{loc}^2$  (and actually even weaker): for any test function  $g(x)$ ,

$$|e^{t[(\nabla - iA(x))^2 - V]}g| \leq e^{t(\Delta - V)}|g|.$$

Now, if we choose a sequence  $g_\epsilon = \epsilon^{-n}g_1(x/\epsilon)$  of (positive) test functions converging to a Dirac delta, we apply the estimate to the test functions  $g = g_\epsilon$  translated at the point  $y$ , and let  $\epsilon \rightarrow 0$ , we obtain an analogous pointwise inequality for the corresponding heat kernels:

$$|e^{t[(\nabla - iA(x))^2 - V]}(x, y)| \leq e^{t(\Delta - V)}(x, y)$$

and (3.27) follows.  $\square$

## 4 Examples of applications of the Main Theorem

In this section we list several applications of the main result Theorem 1.1. They are obtained by combining in a suitable way the Gross type or super log-Sobolev inequalities examined in the previous sections. It is clear that many more examples can be constructed by this procedure, but we decided to focus on the most interesting ones. Notice that some of our examples involve hypoelliptic operators, however our proofs never use this property.



1. *Grushin type operators on  $\mathbb{R}^n$ .*

Theorem 1.1 is especially well suited to study operators of *Grushin type*, which can be defined as sums of the squares of  $n$  vector fields  $Y_1, \dots, Y_n$  on  $\mathbb{R}^n$  with the following structure:

$$Y_1 = \frac{\partial}{\partial x_1}, \quad Y_i = \rho_i(x_1, \dots, x_{i-1}) \frac{\partial}{\partial x_i}, \quad i = 2, \dots, n. \quad (4.29)$$

These vector fields are divergence-free, provided the  $\rho_j$ 's are  $C^1$ ; note however that our theory does not require any smoothness of the coefficients and can be applied also in more general situations.

In the following, we consider a few relevant cases which exemplify the main features of the theory; it is clear that further generalizations are possible.

• **Grushin operators with polynomial coefficients.** Consider on  $\mathbb{R}^n$  the vector fields  $Y_1, \dots, Y_n$  with the structure (4.29), and  $\rho_2, \dots, \rho_n$  are functions satisfying the nondegeneracy condition (1.1), where  $N_i = \rho_i$ . Note that if the  $\rho_i$  are smooth, the family  $(Y_1, \dots, Y_n)$  satisfies Hörmander's condition and hence the operator

$$\mathcal{L} = - \sum_{i=1}^n Y_i^2$$

is hypoelliptic (see [BG]). However, even without assuming smoothness, by a direct application of Theorem 1.1 we obtain the following logarithmic Sobolev inequality: for any  $t > 0$ ,

$$\int_{\mathbb{R}^n} f^2 \ln \frac{f^2}{\|f\|_2^2} dx \leq t \int_{\mathbb{R}^n} \mathcal{L} f \cdot f dx - \frac{n}{2} \ln(\pi e^2 t) \|f\|_2^2 - \int_{\mathbb{R}^n} N(x) |f(x)|^2 dx \quad (4.30)$$

where  $N(x) := \frac{1}{2} \sum_{i=2}^n \ln \rho_i^2(x)$  and  $dx$  is the Lebesgue measure on  $\mathbb{R}^n$ .

It is interesting to notice that if the operator  $\mathcal{L}$  has some dilation invariance, then the log-Sobolev inequality (4.30) is stable, in the following sense. Consider for  $\lambda > 0$  the non isotropic dilation on  $\mathbb{R}^n$

$$H_\lambda(x) = (\lambda^{a_1} x_1, \lambda^{a_2} x_2, \dots, \lambda^{a_n} x_n), \quad x \in \mathbb{R}^n$$

with  $a_i > 0$ , and assume that there exists an index  $d > 0$  such that, for any  $\lambda > 0$ ,

$$\mathcal{L}(f \circ H_\lambda) = \lambda^{2d} (\mathcal{L} f) \circ H_\lambda.$$

This is equivalent to assume that for any  $i = 1, \dots, n$ ,  $\lambda > 0$  and  $x \in \mathbb{R}^n$

$$\lambda^{2a_i} \rho_i^2(x) = \lambda^{2d} \rho_i^2(H_\lambda(x)).$$

Then if we replace  $f$  by  $f \circ H_\lambda$  in (4.30), after a change of variables, we obtain that for any  $t > 0$  and  $\lambda > 0$

$$\int_{\mathbb{R}^n} f^2 \ln \frac{f^2}{\|f\|_2^2} dx \leq t \lambda^{2d} \int_{\mathbb{R}^n} \mathcal{L} f \cdot f dx - \frac{n}{2} \ln(e^2 \pi t \lambda^{2d}) \|f\|_2^2 - \int_{\mathbb{R}^n} N |f|^2 dx.$$

Now it is clear that by setting  $s = t \lambda^{2d}$  the previous inequality reduces to (4.30), i.e. (4.30) is stable.

• **A special Grushin operator in dimension 2.** The following Grushin operator on  $\mathbb{R}^2$  has been extensively studied

$$\mathcal{L} = - \left( \frac{\partial^2}{\partial x^2} + 4x^2 \frac{\partial^2}{\partial y^2} \right).$$

For instance, see [Bec] for the Sobolev inequality which can be related to our subject. The associated vector fields  $Y_1 = \frac{\partial}{\partial x}$  and  $Y_2 = 2x \frac{\partial}{\partial y}$  generate a 2-step nilpotent Lie algebra since  $[Y_1, Y_2] = 2 \frac{\partial}{\partial y}$  and the other brackets are zero. Moreover, the family  $\{X_1, X_2\}$  satisfies the Hörmander condition hence  $\mathcal{L}$  is hypoelliptic.

Denoting the Lebesgue measure on  $\mathbb{R}^2$  by  $d\mu$ , our modified log-Sobolev inequality reads as follows: for any  $t > 0$ ,

$$\int_{\mathbb{R}^2} f^2 \ln \frac{f^2}{\|f\|_2^2} d\mu \leq t \int_{\mathbb{R}^2} \mathcal{L} f \cdot f d\mu + M(t) \|f\|_2^2 - \int_{\mathbb{R}^2} \ln |x| f^2 d\mu \quad (4.31)$$

with  $M(t) := -\ln(2\pi e^2 t)$ . Note that:

- (a) If  $\int_{\mathbb{R}^2} \ln |x| f^2 d\mu = 0$ , the inequality (4.31) becomes similar to the log-Sobolev inequality on  $\mathbb{R}^2$  for the usual Laplacian.
- (b) The operator  $\mathcal{L}$  arises as the induced sub-Laplacian from  $X^2 + Y^2$  on the quotient of the Heisenberg group of dimension 3 by the closed non normal subgroup generated by the vector field  $X$ ; here  $X, Y, Z$  are the usual left-invariant vector fields of the Heisenberg group. This quotient can be identified with  $\mathbb{R}^2$  and has the structure of a nilmanifold [CCFI, p.265]
- (c) The super log-Sobolev inequality (4.31) with fixed  $t_0 > 0$  such that  $M(t_0) = 0$  is a Gross type inequality satisfied by the operator  $A = t_0 \mathcal{L} + V$  where  $V$  is the singular potential  $V(x, y) = -\ln |x|$ . In fact, we can produce many examples of this type with the basic super log-Sobolev inequality (3.16) in the Euclidean space with the Lebesgue measure, see for instance (4.30).
- (d) The log-Sobolev inequality (4.31) is stable by the dilation  $H_\lambda(x, y) = (\lambda x, \lambda^2 y)$  in the sense given at the beginning of the section.

In Section 5, we shall deduce from (4.31) the following log-Sobolev inequality:

$$\int_{\mathbb{R}^2} f^2 \ln \frac{f^2}{\|f\|_2^2} dx dy \leq t \int_{\mathbb{R}^2} \mathcal{L}f \cdot f dx dy + M_1(t) \|f\|_2^2$$

where  $M_1(t) = \ln(c_0 t^{-3/2})$ . This will be obtained by estimating the rightmost expression of (4.31) via a version of Hardy's inequality valid for logarithmic weights, see Lemma 5.2. The same method can be extended to operators of the form

$$\mathcal{L} = - \left( \frac{\partial^2}{\partial x^2} + (x^2)^m \frac{\partial^2}{\partial y^2} \right), \quad m > 0.$$

In [Wa2, Section 4], log-Sobolev with parameter or Gross type inequalities are proved for the quadratic form  $\int \Gamma(f) d\mu_V$ , where  $\Gamma(f) = |\frac{\partial f}{\partial x}|^2 + x^{2m} |\frac{\partial f}{\partial y}|^2$ ,  $m \in \mathbb{N}$ , and  $d\mu_V = e^{-V} dx$  with some potential  $V$  related to quasi-metrics. These examples can be used as a factor  $X_i$  in our semi-direct product theory.

• **A very degenerate Grushin operator.** In [FL] the authors study a very degenerate diffusion on  $\mathbb{R}^2$  of the form

$$\mathcal{L} = -\frac{1}{2} \frac{\partial^2}{\partial x^2} - \frac{1}{2} g^2(x) \frac{\partial^2}{\partial y^2}$$

where  $g(x) > c'_1 \exp(-\frac{c'_2}{|x|^\alpha})$  with  $\alpha \in [0, 2[$ . In particular when  $\alpha \in (0, 2)$  they prove the following pointwise estimate of the heat kernel  $p_t(x, y)$  associated with  $\mathcal{L}$ : for any  $\gamma' > \gamma := \frac{\alpha}{2-\alpha}$ ,

$$p_t(x, y) \leq \frac{c_1}{t} \exp\left(\frac{c_2}{t^{\gamma'}}\right), \quad 0 < t < 1. \quad (4.32)$$

Directly from this estimate one obtains the following log-Sobolev inequality valid for  $0 < t < 1$  :

$$\int_{\mathbb{R}^2} f^2 \ln \frac{f^2}{\|f\|_2^2} d\mu \leq t \int_{\mathbb{R}^2} \mathcal{L}f \cdot f d\mu + M(t) \|f\|_2^2$$

with  $M(t) = c_2 2^{\gamma'} t^{-\gamma'} - \ln t + \ln(2 c_1)$ ; here  $d\mu = dx dy$  is the standard Lebesgue measure. (Notice that in [FL] the exponent  $\gamma'$  is not clearly defined when  $t$  is large).

We can apply our results in this situation and reverse the process, proving a log-Sobolev inequality with parameter first, and deducing as a corollary an upper bound on the heat kernel. Indeed, from Theorem 1.1 we obtain directly for any  $t > 0$

$$\int_{\mathbb{R}^2} f^2 \ln \frac{f^2}{\|f\|_2^2} d\mu \leq t \int_{\mathbb{R}^2} \mathcal{L}f \cdot f d\mu - \ln(\pi e^2 c'_1 2^{-1} t) \|f\|_2^2 + c'_2 \int_{\mathbb{R}^2} \frac{1}{|x|^\alpha} |f|^2 d\mu.$$

In Section 6, we shall prove again a Hardy type inequality which can be applied to the rightmost expression of this inequality, see Lemma 5.3. This will allow to improve the above bound on  $M(t)$  to  $\gamma' = \gamma = \frac{\alpha}{2-\alpha}$ , and also for all positive  $t > 0$ , provided  $\alpha \in (0, 1)$ . Consequently (4.32) holds true with  $\gamma' = \gamma := \frac{\alpha}{2-\alpha}$  and for any  $t > 0$  improving the result of [FL] in the case  $0 < \alpha < 1$ . See Section 5 for the full details. Notice however that our method fails when  $\alpha > 1$ , indeed for any smooth function  $f(x, y)$  which does not vanish near the origin one has  $\int_{\mathbb{R}^2} \frac{1}{|x|^\alpha} |f(x, y)|^2 d\mu = \infty$  and this is an obstruction for the validity of Hardy's inequality. In connection with this, it might be interesting to notice that the operator  $\mathcal{L}$  is hypoelliptic when  $\alpha < 1$ , but it is not hypoelliptic in general when  $\alpha \geq 1$ , and counterexamples depend on the behaviour of  $g$  near the origin.

## 2. Semiproduct of electromagnetic Schrödinger operators.

On  $\mathbb{R}_x^m \times \mathbb{R}_y^n$ , consider the following operator

$$\mathcal{L} = H + a^2(x) K, \quad (4.33)$$

where  $H$  and  $K$  are two *electromagnetic* Schrödinger operators of the form

$$H = (i\nabla_x - A(x))^2 + V(x), \quad K = (i\nabla_y - B(y))^2 + W(y)$$

with suitable potentials  $V$  and  $W$  and  $A : \mathbb{R}^m \rightarrow \mathbb{R}^m$ ,  $B : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , satisfying the assumptions of Proposition 3.2 and  $a^2(x) > 0$ . We recall that as a consequence of Proposition 3.2, we have proved the logarithmic Sobolev inequality with parameter

$$\int_{\mathbb{R}^m} f^2 \ln \frac{f^2}{\|f\|_2^2} dx \leq 2t_0 (Hf, f)_{L^2} + M_0(t_0) \|f\|_2^2$$

where

$$M_0(t_0) = \ln \left( \frac{(2\pi t_0)^{-m/4}}{(1 - \|V_-\|_{K/c_m})^{1/2}} \right)$$

and

$$\int_{\mathbb{R}^n} f^2 \ln \frac{f^2}{\|f\|_2^2} dy \leq 2t_1 (Kf, f)_{L^2} + M_1(t_1) \|f\|_2^2$$

where

$$M_1(t_1) = \ln \left( \frac{(2\pi t_1)^{-n/4}}{(1 - \|W_-\|_{K/c_n})^{1/2}} \right)$$

(see (3.28)). Thus, applying Theorem 1.1, we obtain the following logarithmic Sobolev inequality with parameter: for any  $h(x, y) \in \mathcal{D}(\mathcal{L})$  we have

$$\int h^2 \ln \frac{h^2}{\|h\|_2^2} dx dy \leq t_0 \int H h \cdot h dx dy + t_1 \int a^2(x) K h \cdot h dx dy + \int M(t) h^2 dx dy,$$

where

$$M(t) = M_0(t_0) + M_1(t_1 a^2(x)).$$

It is clear that this result can be generalized to the product of a finite number of spaces and to more general operators.

### 3. Operators on metabelian groups and Hyperbolic spaces.

Let  $\mathbb{H}^{Q+1} = \mathbb{R} \times_t \mathbb{R}^Q$  be the semi-direct product of  $\mathbb{R}$  with  $\mathbb{R}^Q$  defined by the action  $t_a(n) := e^{-a}n$  for  $n \in \mathbb{R}^Q$  and  $a \in \mathbb{R}$ . The product law is given by

$$(a_1, n_1)(a_2, n_2) = (a_1 + a_2, e^{a_2}n_1 + n_2).$$

Consider the right-invariant vector fields

$$Y_0 = \frac{\partial}{\partial a}, \quad Y_i = e^a \frac{\partial}{\partial n_i}, \quad i = 1, \dots, Q.$$

We define the full Laplacian by

$$\mathcal{L} = - \sum_{i=0}^Q Y_i^2$$

and we consider the action of  $\mathcal{L}$  on  $L^2(G)$  with respect to the left-invariant measure  $dx = dadn$  on  $G$  (which coincides with the Lebesgue measure).

The vector fields  $\{Y_0, Y_1, \dots, Y_Q\}$  generate a 2-step solvable Lie algebra since  $[Y_0, Y_i] = Y_i$ ,  $[Y_i, Y_j] = 0$ ,  $i, j = 1, \dots, Q$ , but not nilpotent because  $ad^k(Y_0)(Y_i) = Y_i$  for any  $k \in \mathbb{N}$ . Moreover, the family  $\{Y_0, Y_1, \dots, Y_Q\}$  satisfies the Hörmander condition so  $\mathcal{L}$  is hypoelliptic.

By a direct application of the main result Theorem 1.1 we obtain: for all  $t > 0$

$$\int_{\mathbb{H}^{Q+1}} f^2 \ln \frac{f^2}{\|f\|_2^2} dx \leq t \int_{\mathbb{H}^{Q+1}} \mathcal{L}f \cdot f dx + M(t) \|f\|_2^2 - Q \int_{\mathbb{H}^{Q+1}} a |f(a, n)|^2 dadn$$

where  $M(t) := -\frac{(Q+1)}{2} \ln(e^2 \pi t)$ .

See [H] for other operators in the same class to which we can apply our theory. Note that if for all  $n$  the function  $a \mapsto a|f(a, n)|^2$  is odd and integrable, then the rightmost expression is zero and the super log-Sobolev inequality is formally identical to the same inequality on  $\mathbb{R}^{Q+1}$  for the usual Laplacian.

Differently from the cases of the Grushin operator and the very degenerate Grushin type operator studied above, a Hardy type inequality cannot hold for the term

$$- \int_{\mathbb{R}^{Q+1}} a |f(a, n)|^2 dadn$$

as it is easy to prove. Indeed, suppose that there are  $t > 0$  and  $R(t) \in \mathbb{R}$  such that, for all  $f \in C_0^\infty(\mathbb{R}^{Q+1})$ ,

$$-\int_{\mathbb{R}^{Q+1}} a|f|^2 dadn \leq t \int_{\mathbb{R}^{Q+1}} \left( \left| \frac{\partial f}{\partial a} \right|^2 + e^{2a} |\nabla_n f|^2 \right) dadn + R(t) \int_{\mathbb{R}^{Q+1}} |f|^2 dadn.$$

Fix  $g \in C_0^\infty(\mathbb{R}^{Q+1})$ ,  $g \neq 0$ . Set  $f(a, n) = g(a + k, n)$  with  $k \in \mathbb{N}$  in the preceding inequality and make the change of variables  $u = a + k$ , we get

$$\begin{aligned} -\int_{\mathbb{R}^{Q+1}} u|f|^2 dudn + k \int_{\mathbb{R}^{Q+1}} |f|^2 dudn &\leq t \int_{\mathbb{R}^{Q+1}} \left| \frac{\partial f}{\partial u} \right|^2 dudn + \\ &e^{-2kt} \int_{\mathbb{R}^{Q+1}} e^{2u} |\nabla_n f|^2 dudn + R(t) \int_{\mathbb{R}^{Q+1}} |f|^2 dudn, \end{aligned}$$

which cannot hold when  $k$  goes to infinity, and we get a contradiction as claimed.

#### 4. Products of Damek-Ricci spaces.

Consider a Riemannian manifold  $X$  which is the product of two Damek-Ricci spaces  $S_j$ ,  $j = 0, 1$ :

$$X = S_0 \times S_1$$

and an operator  $\mathcal{L}$  on  $X$  defined as

$$\mathcal{L} = \mathcal{L}_0 + a^2(x_0) \mathcal{L}_1, \quad (4.34)$$

where  $\mathcal{L}_j = -\Delta_{S_j}$  is the Laplace-Beltrami operator on  $S_j$  and  $a^2(x_0) > 0$  is a function depending only on the variable  $x_0 \in S_0$ . Denote by  $Q_j$  the homogeneous dimension of  $S_j$ . In Example 4 of Section 3.2 we have proved the logarithmic Sobolev inequalities with parameter, for  $j = 0, 1$ ,

$$\begin{aligned} \int_{S_j} f^2 \ln \frac{f^2}{\|f\|_2^2} dx_j &\leq t_j (\mathcal{L}_j f, f)_{L^2} + M_j(t_j) \|f\|_2^2 \\ M_j(t) &= \begin{cases} \ln(C_j 2^{n_j/2}) - \frac{n_j}{2} \ln t - \frac{Q_j^2 t}{8} & \text{if } 0 < t \leq 1, \\ \ln(C_j 2^{3/2}) - \frac{3}{2} \ln t - \frac{Q_j^2 t}{8} & \text{if } t > 1. \end{cases} \end{aligned}$$

Thus, applying Theorem 1.1, we have the following logarithmic Sobolev inequality with parameters on  $X = S_0 \times S_1$ : for any  $h(x_0, x_1) \in \mathcal{D}(\mathcal{L})$ ,

$$\begin{aligned} \int_X h^2 \ln \frac{h^2}{\|h\|_2^2} dx_0 dx_1 &\leq t_0 \int_X \mathcal{L}_0 h \cdot h dx_0 dx_1 + t_1 \int_X a^2(x_0) \mathcal{L}_1 h \cdot h dx_0 dx_1 \\ &+ \int_X M(t) h^2 dx_0 dx_1, \end{aligned}$$

where

$$M(t) = M_0(t_0) + M_1(t_1 a^2(x)).$$

## 5. Change of variables

The class of semi-direct product operators is not stable by change of variables. However, we can take advantage of this fact in order to prove super log-Sobolev inequalities for operators which do not belong to this class, as we shall see in the following example. Let us start with a formal proposition.

**Proposition 4.1.** *Let  $M_1, M_2$  be two differentiable manifolds,  $\mu_i$  a measure on  $M_i$  and  $\mathcal{L}_i$  an operator on  $M_i$  for  $i = 1, 2$ . Let  $\Phi : M_1 \rightarrow M_2$  be a  $C^\infty$  diffeomorphism. We assume that*

$$\Phi_*(\mathcal{L}_1) = \mathcal{L}_2, \quad \Phi^*(\mu_1) = \mu_2,$$

*and that  $(M_1, \mathcal{L}_1, \mu_1)$  satisfies a super log-Sobolev inequality of the following form: for any  $f \in C_0^\infty(M_1)$  and any  $t > 0$ ,*

$$\int_{M_1} f^2 \ln \frac{f^2}{\|f\|_2^2} d\mu_1 \leq t \int_{M_1} \mathcal{L}_1 f \cdot f d\mu_1 + \int_{M_1} B_1(t, x) f^2(x) d\mu_1(x). \quad (4.35)$$

*Then  $(M_2, \mathcal{L}_2, \mu_2)$  satisfies a super log-Sobolev inequality of the same form, namely: for any  $g \in C_0^\infty(M_2)$ , and any  $t > 0$ ,*

$$\int_{M_2} g^2 \ln \frac{g^2}{\|g\|_2^2} d\mu_2 \leq t \int_{M_2} \mathcal{L}_2 g \cdot g d\mu_2 + \int_{M_2} B_2(t, y) g^2(y) d\mu_2(y) \quad (4.36)$$

*with  $B_2(t, \Phi(x)) = B_1(t, x)$ ,  $x \in M_1$ .*

In the above statement,  $\Phi_*(\mathcal{L}_1) = \mathcal{L}_2$  and  $\Phi^*(\mu_1) = \mu_2$  mean respectively: for any  $g \in C_0^\infty(M_2)$ ,

$$(\mathcal{L}_2 g) \circ \Phi = \mathcal{L}_1(g \circ \Phi)$$

and

$$\int_{M_1} (g \circ \Phi) d\mu_1 = \int_{M_2} g d\mu_2.$$

**Proof:** Let  $g \in C_0^\infty(M_2)$  and set  $f = g \circ \Phi$  so that  $f \in C_0^\infty(M_1)$ . Putting this function  $f$  in (4.35) we deduce immediately (4.36).

To illustrate the interest of Proposition 4.1, we apply it to the operator

$$\mathcal{L}_1 f = -\frac{\partial^2 f}{\partial^2 a} - e^{2a} \sum_{i=1}^Q \frac{\partial^2 f}{\partial^2 n_i}$$

on the space  $M_1 = \mathbb{H}^{Q+1}$  with the Lebesgue measure  $d\mu_1 = dadn$ . The change of variables  $\Phi : \mathbb{R}^{Q+1} \rightarrow (0, +\infty) \times \mathbb{R}^Q$  is given by  $\Phi(a, n) = (e^a, n) = (r, n)$ .

Then, starting from

$$\begin{aligned} \int_{\mathbb{R}^{Q+1}} f^2 \ln \frac{f^2}{\|f\|_2^2} dadn &\leq t \int_{\mathbb{R}^{Q+1}} \mathcal{L}_1 f \cdot f dadn - Q \int_{\mathbb{R}^{Q+1}} af^2(a, n) dadn \\ &\quad - \frac{Q+1}{2} \ln(\pi e^2 t) \|f\|_2^2, \end{aligned} \quad (4.37)$$

we get for the transformed operator

$$\mathcal{L}_2 = -r^2 \left( \frac{\partial^2}{\partial^2 r} + \sum_{i=1}^Q \frac{\partial^2}{\partial^2 n_i} \right)$$

the modified super log-Sobolev inequality

$$\begin{aligned} \int_{(0, \infty) \times \mathbb{R}^Q} g^2 \ln \frac{g^2}{\|g\|_2^2} \frac{dr}{r} dn &\leq t \int_{(0, \infty) \times \mathbb{R}^Q} \mathcal{L}_2 g \cdot g \frac{dr}{r} dn \\ &\quad - Q \int_{(0, \infty) \times \mathbb{R}^Q} (\ln r) g^2(r, n) \frac{dr}{r} dn - \frac{Q+1}{2} \ln(\pi e^2 t) \|g\|_2^2 \end{aligned}$$

where  $d\mu_2(r, n) = \frac{dr}{r} dn$ .

Thus we see that, although the transformed operator is not a semidirect product and we can not apply our theory directly, nevertheless we are able to prove a super log-Sobolev inequality. The modified super log-Sobolev inequality just proved, when  $Q = 1$ , can be interpreted as a modified super log-Sobolev inequality for the Laplace-Beltrami operator  $\Delta = -\mathcal{L}_2$  on the Poincaré upper half-plane  $\mathbf{H}$  with respect to the weighted Riemannian measure  $d\mu(r, n) = \omega(r)r^{-2}drdn$  where  $r^{-2}drdn$  is the Riemannian measure on  $\mathbf{H}$  and the weight  $\omega(r) = r$ .

Another interesting application can be obtained by using as change of variables the family of diffeomorphisms given by the left-invariant actions  $\Phi_g(h) = gh$ . This idea produces super log-Sobolev inequalities for a whole family of second order differential operators with drift parametrized by elements of the group.

For simplicity we focus on the case  $Q = 1$ . Then, with the notations  $g = (a_1, n_1)$  and  $h = (a, n)$ , we can write explicitly  $\Phi_g(h) = (a_1 + a, e^{a_1}n_1 + n)$ . Replacing  $f$  by  $f \circ \Phi_g$  in the super log-Sobolev inequality and recalling that the measure is left-invariant, we arrive at

$$\begin{aligned} \int_{\mathbb{R}^2} f^2 \ln \frac{f^2}{\|f\|_2^2} dadn &\leq t \int_{\mathbb{R}^2} \mathcal{L}^{a_1, n_1} f \cdot f dadn - \int_{\mathbb{R}^2} af^2(a, n) dadn \\ &\quad + (a_1 - \ln(e^2 \pi t)) \|f\|_2^2 \end{aligned}$$

where

$$\mathcal{L}^{a_1, n_1} = Y_0^2 + (n_1^2 + 1)e^{-2a_1}Y_1^2 + 2n_1e^{-a_1}Y_1Y_0 + n_1e^{-a_1}Y_1$$



and  $Y_0 = \partial_a$ ,  $Y_1 = e^a \partial_n$  and  $\partial_n$  is the derivative with respect to the variable  $n$ .

Each operator  $\mathcal{L}^{a_1, n_1}$  is a symmetric second-order linear differential operator with drift on  $L^2(dadn)$ . Note that the super log-Sobolev inequality (4.37) is invariant by the right-action of the group, thus if we use instead the right-invariant actions, we do not obtain new inequalities. To check invariance we notice that the vector fields are right-invariant, the Jacobian determinant equals to the constant  $1/\Delta(g) = e^{-Qa_1}$  where the modular function  $\Delta$  satisfies the relation  $\Delta(g) \int f^2(xg) dx = \int f^2(x) dx$ ; then we use the fact that  $\ln \Delta$  is linear and the relation  $-N + \ln \Delta = 0$  with  $N$  as in (4.30).

## 5 Application to ultracontractive bounds

It is well known that from the super log-Sobolev inequality one can deduce suitable ultracontractive bounds and estimates on the heat kernel. Moreover, the Gross inequality implies the concentration phenomenon when the reference measure is a probability measure (but we shall not go that direction in this paper). In this section, we focus on the first application starting from our modified log-Sobolev inequality (1.4) in some simple cases. But we need an additional tool, namely Hardy type inequalities. The general statement is the following:

**Theorem 5.1.** *Let  $(X, \mu)$  be a manifold with a positive measure  $\mu$ . Let  $A$  be a non-negative selfadjoint operator on  $L^2(X, \mu)$  which generates a semigroup  $e^{-tA}$ . Assume that*

1.  $C_0^\infty(X) \subset \mathcal{D}(A)$ .
2. *For any  $t > 0$  and any  $h \in C_0^\infty(X)$ , we have the following modified log-Sobolev inequality*

$$\int_X h^2 \ln \frac{h^2}{\|h\|_2^2} d\mu \leq t(Ah, h) + \ln(c_0 t^{-n/2}) \|h\|_2^2 - \int_X Nh^2 d\mu \quad (5.38)$$

*for some  $c_0, n > 0$  and some function  $N : X \rightarrow \mathbb{R}$ .*

3. *For any  $t > 0$  and any  $h \in C_0^\infty(X)$ , we have a Hardy type inequality*

$$- \int_X Nh^2 d\mu \leq t(Ah, h) + g(t) \|h\|_2^2 \quad (5.39)$$

*for some function  $g : (0, +\infty) \rightarrow \mathbb{R}$ .*

Then

$$\|T_t\|_{2 \rightarrow \infty} \leq c_1 t^{-n/4} e^{M(t)}$$

with  $M(t) = (2t)^{-1} \int_0^t g(\varepsilon) d\varepsilon$  and some constant  $c_1 > 0$ .

Note that what we call a Hardy type inequality (5.39) is similar to the assumption (4.4.2) in Rosen's Lemma 4.4.1 of [D] (provided we choose  $N = \ln \Phi$ ). The purpose of such a condition is to get a true log-Sobolev inequality with parameter which allows us to apply Corollary 2.2.8 of [D].

**Proof:** From (5.38) and (5.39), we get for any  $t > 0$  and any  $h \in C_0^\infty(X)$ ,

$$\int_X h^2 \ln \frac{|h|}{\|h\|_2} d\mu \leq t(Ah, h) + \left( \ln(c_0^{1/2} t^{-n/4}) + 2^{-1}g(t) \right) \|h\|_2^2.$$

Now it is sufficient to apply Corollary 2.2.8 of [D] to conclude the proof.

We give two applications of this result.

1. Let  $m > 0$ . The generalized Grushin operator on  $\mathbb{R}^2$

$$\mathcal{L} = - \left( \frac{\partial}{\partial x} \right)^2 - (x^2)^m \left( \frac{\partial}{\partial y} \right)^2$$

which is a particular case of Example 1 of Section 4, satisfies the inequality

$$\int_{\mathbb{R}^2} f^2 \ln \frac{f^2}{\|f\|^2} d\mu \leq t \|\nabla_{\mathcal{L}} f\|_2^2 - \ln(\pi e^2 t) \|f\|_2^2 - m \int_{\mathbb{R}^2} \ln |x| f^2(x, y) d\mu$$

for any  $t > 0$  where  $d\mu = dx dy$ . In order to apply Theorem 5.1, we shall need the following Hardy type inequality:

**Lemma 5.2.** *For any  $t > 0$ ,*

$$- \int_{\mathbb{R}^2} \ln |x| f^2(x, y) dx dy \leq t \|\nabla_{\mathcal{L}} f\|_2^2 + \frac{1}{2} \ln(2et^{-1}) \|f\|_2^2.$$

Actually this lemma will be proved with  $\|\frac{\partial f}{\partial x}\|_2^2$  instead of  $\|\nabla_{\mathcal{L}} f\|_2^2$ ; the above formulation follows from the trivial inequality  $\|\frac{\partial f}{\partial x}\|_2^2 \leq \|\nabla_{\mathcal{L}} f\|_2^2$ . We postpone the proof to the Appendix, see Section 6.1.

By Theorem 5.1, the two last inequalities imply that, for any  $t > 0$ ,

$$\int_{\mathbb{R}^2} f^2 \ln \frac{|f|}{\|f\|_2} dx dy \leq t \|\nabla_{\mathcal{L}} f\|_2^2 + \ln \left( k t^{-\frac{1}{2} - \frac{m}{4}} \right) \|f\|_2^2$$

for a suitable  $k > 0$ . Then by Corollary 2.2.8 in [D], for any  $t > 0$ , we deduce

$$\|T_t f\|_{2 \rightarrow \infty} \leq k' t^{-\frac{1}{2} - \frac{m}{4}}$$

and this implies the following uniform bound for the heat kernel  $h_t$  of  $\mathcal{L}$ : for any  $p = (x, y) \in \mathbb{R}^2$  and for any  $t > 0$ ,

$$h_t(p, p) \leq k'' t^{-1 - \frac{m}{2}}.$$

Such a result is known only in the case  $m = 1$ , at least to the authors' knowledge; note that here  $m$  is not necessarily an integer.

Moreover, using the expression of the heat kernel on the diagonal in the case  $m = 1$  obtained by G. Ben Arous (see [Ben-Ar, Eq.(1.20)]) via a probabilistic proof, we can see that our result is sharp.

2. We consider the following very degenerate Grushin operator (see [FL]) defined on  $\mathbb{R}^2$  by

$$\mathcal{L} = - \left( \frac{\partial}{\partial x} \right)^2 - \exp \left( - \frac{2}{|x|^\alpha} \right) \left( \frac{\partial}{\partial y} \right)^2,$$

(see Example 1 of Section 4). By applying the inequality (4.30), we get for any  $t > 0$ ,

$$\int_{\mathbb{R}^2} f^2 \ln \frac{f^2}{\|f\|^2} dx dy \leq t \|\nabla_{\mathcal{L}} f\|_2^2 - \ln(\pi e^2 t) \|f\|_2^2 + \int_{\mathbb{R}^2} \frac{1}{|x|^\alpha} f^2(x, y) dx dy.$$

The following Hardy type inequality holds (proved below in Section 6.2):

**Lemma 5.3.** *For any  $0 < \alpha < 1$ , there exist  $c > 0$  such that, for any  $t > 0$ ,*

$$\int_{\mathbb{R}^2} \frac{1}{|x|^\alpha} f^2(x, y) dx dy \leq t \|\nabla_{\mathcal{L}} f\|_2^2 + c t^{-b} \|f\|_2^2$$

with  $b = \frac{\alpha}{2-\alpha}$  (and  $b$  is the unique exponent satisfying this inequality above).

The remark after Lemma 5.2 applies in a similar way here for the gradient  $\nabla_{\mathcal{L}}$ . Then by Theorem 5.1, the two last inequalities imply, for any  $t > 0$ ,

$$\int_{\mathbb{R}^2} f^2 \ln \frac{|f|}{\|f\|_2} dx dy \leq t \|\nabla_{\mathcal{L}} f\|_2^2 + \left( -\frac{1}{2} \ln(\pi e^2 t) + \frac{c}{2} t^{-b} \right) \|f\|_2^2.$$

Now, following Example 2.3.4 of [D], we conclude that for any  $t > 0$ ,

$$\|T_t\|_{2 \rightarrow \infty} \leq k' t^{-\frac{1}{2}} \exp(c' t^{-b})$$

which implies the uniform bound on the heat kernel  $h_t$  of  $\mathcal{L}$ , for any  $p = (x, y) \in \mathbb{R}^2$  and any  $t > 0$ ,

$$h_t(p, p) \leq k'' t^{-1} \exp(c'' t^{-b}).$$

Additional results related to this operator can be found in [FL].

## 6 Appendix: Hardy type Lemmas

This appendix is devoted to the proof of the lemmas stated and used in Section 5.

## 6.1 Proof of Lemma 5.2

Since

$$\left\| \frac{\partial f}{\partial x} \right\|_{L^2(\mathbb{R}^2)} \leq \|\nabla \mathcal{L} f\|_{L^2(\mathbb{R}^2)},$$

we see that it is sufficient to prove the one dimensional estimate

$$\int_{\mathbb{R}} (-\ln |x|) f^2(x) dx \leq t \|f'\|_2^2 + \frac{1}{2} \ln(2et^{-1}) \|f\|_2^2, \quad t > 0. \quad (6.40)$$

where the norms are now in  $L^2(\mathbb{R})$ . The proof of (6.40) will descend from the following proposition:

**Proposition 6.1.** *For all  $0 < \delta \leq 1$*

$$\int_0^1 -\ln x \cdot |f|^2 dx \leq |\ln \delta| \cdot \|f\|_{L^2(0,1)}^2 + \|f\|_{L^2(0,\delta)}^2 + \frac{2\delta}{e} \|f\|_{L^2(0,\delta)} \|f'\|_{L^2(0,\delta)}. \quad (6.41)$$

*Proof.* It is sufficient to prove the estimate for a smooth function  $f$ . We have the identity

$$\int_0^1 (-\ln x) |f|^2 dx = \int_0^1 (-\ln x) \frac{d}{dx} \int_0^x |f|^2 = \int_0^1 \frac{1}{x} \int_0^x |f|^2 + (-\ln x) \int_0^x |f|^2 \Big|_0^1$$

and we notice that the last boundary term vanishes. We estimate the remaining integral at the r.h.s. as follows. The piece of the integral with  $\delta \leq x \leq 1$  is bounded by

$$\int_{\delta}^1 \frac{1}{x} \int_0^x |f|^2 d\xi dx \leq \|f\|_{L^2(0,1)}^2 \int_{\delta}^1 \frac{1}{x} dx = \|f\|_{L^2(0,1)}^2 \cdot |\ln \delta|. \quad (6.42)$$

On the other hand, an integration by parts gives:

$$\begin{aligned} \int_0^{\delta} \frac{1}{x} \int_0^x |f|^2 d\xi dx &= \int_0^{\delta} \frac{1}{x} \int_0^x \xi' |f|^2 d\xi dx = \int_0^{\delta} \frac{1}{x} \left[ -\int_0^x 2\xi f f' d\xi + \xi |f|^2 \Big|_0^x \right] dx \\ &= \int_0^{\delta} \int_0^x \left( -\frac{2\xi}{x} \right) f f' d\xi dx + \int_0^{\delta} |f|^2 dx. \end{aligned} \quad (6.43)$$

Now we notice that

$$\begin{aligned} \int_0^{\delta} \int_0^x \left( -\frac{2\xi}{x} \right) f f' d\xi dx &= \int_0^{\delta} \int_{\xi}^{\delta} \left( -\frac{2\xi}{x} \right) f f' dx d\xi = \\ &= -2 \int_0^{\delta} f f' \left( \xi \ln \frac{\delta}{\xi} \right) d\xi. \end{aligned}$$

The function  $\xi \ln(\delta/\xi)$  is non negative on  $[0, \delta]$  and vanishes at the boundary; its maximum is at  $\xi = \delta/e$  so that

$$\left| \xi \ln \frac{\delta}{\xi} \right| \leq \frac{\delta}{e}.$$

This implies

$$\left| \int_0^\delta \int_0^x \left( -\frac{2\xi}{x} \right) f f' d\xi dx \right| \leq \frac{2\delta}{e} \|f\|_{L^2(0,\delta)} \|f'\|_{L^2(0,\delta)}$$

and by (6.43)

$$\int_0^\delta \frac{1}{x} \int_0^x |f|^2 d\xi dx \leq \frac{2\delta}{e} \|f\|_{L^2(0,\delta)} \|f'\|_{L^2(0,\delta)} + \|f\|_{L^2(0,\delta)}^2.$$

Putting this estimate together with (6.42) we obtain (6.41).  $\square$

Now, we are in position to prove Lemma 5.2. By changing  $f(x)$  by  $f(-x)$ , we get from (6.41)

$$\int_{-1}^0 -\ln|x| \cdot |f|^2 dx \leq |\ln \delta| \cdot \|f\|_{L^2(-1,0)}^2 + \|f\|_{L^2(-\delta,0)}^2 + \frac{2\delta}{e} \|f\|_{L^2(-\delta,0)} \|f'\|_{L^2(-\delta,0)}. \quad (6.44)$$

By the inequality  $2ab \leq \frac{1}{s}a^2 + sb^2$  valid for any  $s > 0$ , we deduce

$$2\|f\|_{L^2(-\delta,0)} \|f'\|_{L^2(-\delta,0)} + 2\|f\|_{L^2(0,\delta)} \|f'\|_{L^2(0,\delta)} \leq \frac{1}{s} \|f'\|_{L^2(-\delta,\delta)}^2 + s\|f\|_{L^2(-\delta,\delta)}^2.$$

By summing up with (6.41), for any  $s > 0$  and  $\delta \in (0, 1]$ , we have

$$\int_{\mathbb{R}} -\ln|x| \cdot |f|^2 dx \leq \int_{-1}^1 -\ln|x| \cdot |f|^2 dx \leq \frac{\delta}{es} \|f'\|_2^2 + \left( |\ln \delta| + \frac{s\delta}{e} + 1 \right) \|f\|_2^2.$$

We have obtained an inequality of the form

$$\int_{\mathbb{R}} -\ln|x| \cdot |f|^2 dx \leq c_1 \|f'\|_2^2 + c_2 \|f\|_2^2$$

with  $c_i > 0$ . Here, we use a dilation argument by applying this inequality to the rescaled function

$$f(x) = g(x\sqrt{t/c_1}), \quad t > 0,$$

and we obtain

$$\int_{\mathbb{R}} -\ln|x| \cdot |f|^2 dx \leq t \|f'\|_2^2 + \ln \left( \frac{\sqrt{c_1} e^{c_2}}{\sqrt{t}} \right) \|f\|_2^2.$$

Taking  $c_1 = \frac{\delta}{es}$  and  $c_2 = |\ln \delta| + \frac{s\delta}{e} + 1$ ,

$$c(s\delta) := \sqrt{c_1} e^{c_2} = \sqrt{\frac{e}{s\delta}} e^{\frac{s\delta}{e}}.$$

We minimize  $c(s\delta)$  over  $s\delta > 0$  and get  $\inf_{s\delta > 0} c(s\delta) = \inf_{u > 0} \sqrt{\frac{1}{u}} e^u = \sqrt{2e}$ , which implies (6.40) as claimed.

## 6.2 Proof of Lemma 5.3

Let  $0 < \alpha < 1, \delta > 0$  and  $f \in C_0^\infty(\mathbb{R})$ . We write

$$I_\alpha(f) = \int_0^\infty \frac{1}{|x|^\alpha} f^2(x) dx = J_\alpha + K_\alpha$$

with  $J_\alpha = \int_0^\delta \frac{1}{|x|^\alpha} f^2(x) dx$  and  $K_\alpha = \int_\delta^\infty \frac{1}{|x|^\alpha} f^2(x) dx$ . Obviously,  $K_\alpha \leq \delta^{-\alpha} \|f\|_2^2$ . By integration by parts,

$$\begin{aligned} J_\alpha &= \left[ \frac{x^{1-\alpha}}{1-\alpha} f^2(x) \right]_0^\delta - \frac{2}{1-\alpha} \int_0^\delta x^{1-\alpha} f f' dx \\ &\leq \frac{\delta^{1-\alpha}}{1-\alpha} f^2(\delta) + \frac{1}{1-\alpha} \delta^{1-\alpha} \left( \frac{1}{\delta} \|f\|_2^2 + \delta \|f'\|_2^2 \right). \end{aligned}$$

This last inequality comes from:

$$2|f f'| \leq \frac{1}{\delta} f^2 + \delta (f')^2.$$

We now prove

$$|f(\delta)| \leq \frac{1}{\sqrt{\delta}} \|f\|_2 + \sqrt{\delta} \|f'\|_2. \quad (6.45)$$

Let  $x_0 \in [0, \delta]$  such that  $|f(x_0)| = \inf_{[0, \delta]} |f(x)|$ . Then

$$|f(\delta)| \leq |f(\delta) - f(x_0)| + |f(x_0)| \leq \int_0^\delta |f'| + \frac{1}{\sqrt{\delta}} \|f\|_2 \leq \sqrt{\delta} \|f'\|_2 + \frac{1}{\sqrt{\delta}} \|f\|_2$$

by Hölder inequality. We deduce

$$|f(\delta)|^2 \leq 2\delta \|f'\|_2^2 + \frac{2}{\delta} \|f\|_2^2.$$

Therefore,

$$J_\alpha \leq \frac{3}{1-\alpha} (\delta^{2-\alpha} \|f'\|_2^2 + \delta^{-\alpha} \|f\|_2^2).$$

From this bound and the bound on  $K_\alpha$ , we get

$$I_\alpha(f) \leq \frac{3}{1-\alpha} \delta^{2-\alpha} \|f'\|_2^2 + \frac{4-\alpha}{1-\alpha} \delta^{-\alpha} \|f\|_2^2. \quad (6.46)$$

This inequality is stable by dilation. Indeed, changing  $f(x)$  by  $f_\lambda(x) = f(\lambda x)$ , we obtain

$$I_\alpha(f) \leq \frac{3}{1-\alpha} (\delta \lambda)^{2-\alpha} \|f'\|_2^2 + \frac{4-\alpha}{1-\alpha} (\delta \lambda)^{-\alpha} \|f\|_2^2.$$

This reduces to (6.46) by setting  $s = \delta \lambda$ .

We set  $c_1(\alpha) = \frac{3}{1-\alpha}$  and  $c_2(\alpha) = \frac{4-\alpha}{1-\alpha}$ . Let  $t > 0$  and choose  $\delta$  such that  $t = c_1(\alpha)\delta^{2-\alpha}$ . We set  $\gamma = \frac{\alpha}{2-\alpha}$ . The inequality (6.46) is equivalent to

$$I_\alpha(f) \leq t\|f'\|_2^2 + c_3 t^{-\gamma}\|f\|_2^2.$$

with  $c_3 = c_2 c_1^\gamma$ . The  $L^2$ -norm are the norm on  $L^2(\mathbb{R}^+)$ . We easily deduce the result on  $\mathbb{R}$ ,

$$\int_{-\infty}^{\infty} \frac{1}{|x|^\alpha} f^2(x) dx = \int_0^{\infty} \frac{1}{|x|^\alpha} f^2(x) dx + \int_0^{\infty} \frac{1}{|x|^\alpha} f^2(-x) dx \leq t\|f'\|_2^2 + c_3 t^{-\gamma}\|f\|_2^2,$$

where now, the  $L^2$ -norm are the norm on  $L^2(\mathbb{R})$ . To finish the proof of Lemma 5.3, for  $g \in C_0^\infty(\mathbb{R}^2)$  and any  $y \in \mathbb{R}$ , we set  $f(x) = g(x, y)$  in the inequality just above and integrate this inequality over  $\mathbb{R}$  in  $y$ . We obtain

$$\int_{\mathbb{R}^2} \frac{1}{|x|^\alpha} g^2(x, y) dx dy \leq t \left\| \frac{\partial g}{\partial x} \right\|_2^2 + c_3 t^{-\gamma} \|g\|_2^2.$$

We conclude the lemma by the fact that

$$\left\| \frac{\partial g}{\partial x} \right\|_{L^2(\mathbb{R}^2)}^2 \leq (\mathcal{L}g, g).$$

We take  $b = \gamma$  to prove our inequality.

**Proof of uniqueness of  $b$ .** We use a dilation argument. Let  $b' > 0$  such that, for any  $t > 0$ ,

$$\int_{\mathbb{R}^2} \frac{1}{|x|^\alpha} g^2(x, y) dx dy \leq t(\mathcal{L}g, g) + c_3 t^{-b'} \|g\|_2^2.$$

Replace now  $g$  with  $g \circ H_\lambda$  where  $H_\lambda(x, y) = (\lambda x, \lambda^\beta y)$ ,  $\lambda > 0$ , for a fixed  $\beta > 1$ ; after a change of variables, we get for any  $t > 0$  and  $\lambda > 0$ :

$$\int_{\mathbb{R}^2} \frac{1}{|x|^\alpha} g^2(x, y) dx dy \leq t\lambda^{2-\alpha}(\mathcal{L}_\lambda g, g) + c_3 t^{-b'} \lambda^{-\alpha} \|g\|_2^2 \quad (6.47)$$

with

$$\mathcal{L}_\lambda = \mathcal{L}_{\lambda, \beta} := - \left( \frac{\partial}{\partial x} \right)^2 - \lambda^{2\beta-2} \exp \left( -\frac{2\lambda^\alpha}{|x|^\alpha} \right) \left( \frac{\partial}{\partial y} \right)^2.$$

Let  $s > 0$ ,  $\lambda > 0$  and choose  $t > 0$  in (6.47) such that  $s = t\lambda^{2-\alpha}$ , then

$$\int_{\mathbb{R}^2} \frac{1}{|x|^\alpha} g^2(x, y) dx dy \leq s(\mathcal{L}_\lambda g, g) + c_3 s^{-b'} \lambda^{-b'(\alpha-2)-\alpha} \|g\|_2^2.$$

Assume  $b' > b$  and let  $\lambda$  tend to 0 and  $s$  also (in that order), we get

$$\int_{\mathbb{R}^2} \frac{1}{|x|^\alpha} g^2(x, y) dx dy = 0$$

for any function  $g$ : contradiction.

Now, let  $s > 0, \lambda > 0$  and choose  $t > 0$  in (6.47) such that  $s = t^{-b'} \lambda^{-\alpha}$ . Then

$$\int_{\mathbb{R}^2} \frac{1}{|x|^\alpha} g^2(x, y) dx dy \leq s^{-\frac{1}{b'}} \lambda^{-\frac{\alpha}{b'} + 2 - \alpha} (\mathcal{L}_\lambda g, g) + c_3 s \|g\|_2^2.$$

Assume  $b > b'$  and let  $\lambda$  tend to  $+\infty$  and  $s$  tend to 0 (in that order), we get the same contradiction. So  $b' = b$ . The proof is completed.

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